The Series Solution of Problems in the Calculus of Variations via the Homotopy Analysis Method

Saeid Abbasbandy and Ahmand Shirzadi

Department of Mathematics, Imam Khomeini International University, Ghazvin, 34149-16818, Iran

Reprint requests to S. A.; Fax: +98 281 3780040; E-mail: abbasbandy@yahoo.com

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The homotopy analysis method (HAM) is used for solving the ordinary differential equations which arise from problems of the calculus of variations. Some numerical results are given to demonstrate the validity and applicability of the presented technique. The method is very effective and yields very accurate results.

Key words: Calculus of Variations; Euler-Lagrange Equation; Homotopy Analysis Method.
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1. Introduction

Problems that deal with finding minima or maxima of a functional are called variational problems. Several important variational problems such as the brachistochrone problem, the problem of geodesics, and the isoperimetric problem were first posed at the end of the 17th century (beginning in 1696). General methods for solving variational problems were created by L. Euler and J. Lagrange in the 18th century. Later on, the variational calculus became an independent mathematical discipline with its own research methods.

The variational calculus problems can often be transformed into differential equations. Unfortunately, the only problems that can be solved exactly, seem to be the classical problems of mathematical physics whose solutions are already well known.

For solving variational problems, Tatari and Dehghan [1] used the variational iteration method, Abdulaziz et al. [2] used the homotopy perturbation method, Dehghan and Tatari [3] used the Adomian decomposition method, and Saadatmandi and Dehghan [4] used the Chebyshev finite difference method. The direct method of Ritz and Galerkin is well covered in many textbooks [5 – 7]. Chen and Hsiao [8] introduced the Walsh series method to variational problems. See [9 – 12] for using Laguerre polynomials, Legendre polynomials, Chebyshev series, and Legendre wavelet approaches, respectively, to derive continuous solutions for the variational problems. For other methods, the interested reader might see Razzaghi and Razzaghi [13] for the Fourier series direct method, Razzaghi and Ordokhani [14] for rationalized Harr functions, Hsiao [15] for the wavelet direct method, and Glabisz [16] for the direct Walsh, wavelet packet method.

In this paper the hamotopy analysis method (HAM) is used for solving ordinary differential equations which arise from problems of the calculus of variations. This approach is described briefly in Section 3 of this paper.

The HAM [17, 18] was first proposed by Liao in 1992. The HAM was further developed and improved by Liao for nonlinear problems [19], for solving solitary waves with discontinuity [20], for series solutions of nano-boundary layer flows [21], for nonlinear equations [22], and many other subjects [23 – 31].

The application of the HAM in mathematical problems is highly considered by scientists, because the HAM provides us with a convenient way to control the convergence of approximation series which is a fundamental qualitative difference in analysis between the HAM and other methods.

The remaining structure of this article is organized as follows: Section 2 is a brief basic for the calculus of the variation theory. Section 3 briefly reviews the mathematical basis of the HAM used for this study. Two illustrative examples are documented in Section 4. These examples intuitively describe the ability and reliability of the method. A conclusion and future directions for research are summarized in the last section.

2. Basics of the Calculus of Variations

The simplest form of a variational problem can be considered as finding the extremum of the functional

$$v[y] = \int_{x_0}^{x_1} [F(x, y(x), y'(x))] \mathrm{d}x. \tag{1}$$

To find the extreme value of v, the boundary conditions are known in the form

$$y(x_0) = \alpha, \quad y(x_1) = \beta. \tag{2}$$

The necessary condition for the solution of problem (1) is to satisfy the Euler-Lagrange equation

$$F_y - \frac{\mathrm{d}}{\mathrm{d}x} F_{y'} = 0 \tag{3}$$

with the boundary conditions (2). The Euler-Lagrange equation is generally nonlinear. In this work we apply the HAM for solving Euler-Lagrange equations which arise from problems in the calculus of variations. It is shown that this scheme is efficient for solving these kinds of problems.

The boundary value problem (3) does not always have a solution and, if a solution exists, it may not be unique. Note that in many variational problems, from the physical or geometrical meaning of the problem the existence of a solution is obvious and is unique if the solution of Euler's equation satisfies the boundary conditions and if it is the solution of the given variational problem [7].

The general form of the variational problem (1) is

$$\begin{aligned} &v[y_1, y_2, \dots, y_n] = \\ &\int_{x_0}^{x_1} [F(x, y_1, y_2, \dots, y_n, y_1', y_2', \dots, y_n')] \mathrm{d}x \end{aligned} \tag{4}$$

with the given boundary conditions

$$\begin{aligned} &y_1(x_0) = \alpha_1, \quad y_2(x_0) = \alpha_2, \quad \dots y_n(x_0) = \alpha_n, \\ &y_1(x_1) = \beta_1, \quad y_2(x_1) = \beta_2, \quad \dots y_n(x_1) = \beta_n. \end{aligned}$$

Here the necessary condition for the existence of the extremum of the functional (4) is to satisfy the system of second-order differential equations

$$F_{y_i} - \frac{\mathrm{d}}{\mathrm{d}x} F_{y_i'} = 0, \quad i = 1, 2, \dots, n,$$

with the above boundary conditions.

3. The Homotopy Analysis Method

To illustrate the basic concept of the HAM, we consider the general nonlinear system

$$\mathcal{N}[u(x)] = 0,$$

where $\mathcal{N}$ is a nonlinear operator, x denotes an independent variable, and $u(x)$ is an unknown function, respectively. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional homotopy method, Liao constructed the so-called zero-order deformation equation

$$(1-p)\mathcal{L}[\phi(x;p) - u_0(x)] = p\hbar\mathcal{H}(x)\mathcal{N}[\phi(x;p)], \tag{5}$$

where $p \in [0,1]$ is the embedding parameter, $\hbar \neq 0$ is the convergence-control parameter [32], $\mathcal{H}(x) \neq 0$ is an auxiliary function, $\mathcal{L}$ is an auxiliary linear operator, $u_0(x)$ is an initial guess of $u(x)$, and $\phi(x;p)$ is a unknown function, respectively. It is important, that one has great freedom to choose auxiliary parameters in the HAM. Obviously, when $p = 0$ and $p = 1$, it holds

$$\phi(x;0) = u_0(x), \quad \phi(x;1) = u(x),$$

respectively. Thus as p increases from 0 to 1, the solution $\phi(x;p)$ varies from the initial guess $u_0(x)$ to the solution $u(x)$. Expanding $\phi(x;p)$ in a Taylor series with respect to p, one has

$$\phi(x;p) = u_0(x) + \sum_{m=1}^{+\infty} u_m(x) p^m, \tag{6}$$

where

$$u_m(x) = \frac{1}{m!} \frac{\partial^m \phi(x;p)}{\partial p^m}\Big|_{p=0}.$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$, and the auxiliary function are properly chosen, the series (6) converges at $p = 1$ to

$$u(x) = u_0(x) + \sum_{m=1}^{+\infty} u_m(x),$$

which must be one of the solutions of the original nonlinear equation, as proved by Liao. The governing equation can be deduced from the zero-order deformation equation. Defining the vector

$$\vec{u}_n = \{u_0(x), u_1(x), \dots, u_n(x)\},$$

differentiating (5) m times with respect to the embedding parameter p, then setting $p = 0$, and finally dividing by $m!$, we have the so-called mth-order deformation equation

$$\mathcal{L}[u_m(x) - \chi_m u_{m-1}(x)] = \hbar \mathcal{H}(x) R_m(\vec{u}_{m-1}), \quad (7)$$

where

$$R_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} \mathcal{N}[\phi(x;p)]}{\partial p^{m-1}} \Big|_{p=0} \quad (8)$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases}$$

4. The HAM for Problems in the Calculus of Variations

In this section, we present two examples to show the efficiency and high accuracy of the present method for finding the numerical solution of problems in the calculus of variations.

4.1. Example 1

The brachistochrone problem is one of the earliest problems posed in the calculus of variations. It was proposed in 1696 by Johann Bernoulli to find the line connecting two certain points, A and B, that do not lie on a vectorial line and possess the property that a moving particle slides down this line from A to B in the shortest time. This problem was solved by Johann Bernoulli, Jacob Bernoulli, Leibnitz, Newton and L'Hospital. It was shown that the solution of this problem is a cycloid.

Consider the brachistochrone problem [1, 4, 33]

$$\min\ v[y] = \int_0^1 \left[\frac{1 + y'^2(x)}{1 - y(x)} \right]^{\frac{1}{2}} \mathrm{d}x \quad (9)$$

with the boundary conditions

$$y(0) = 0, \quad y(1) = -0.5. \quad (10)$$

In this case the Euler-Lagrange equation is in the following form:

$$y'' = -\frac{1 + y'^2}{2(y-1)}$$

or, equivalent,

$$y'' - yy'' - \frac{y'^2}{2} - \frac{1}{2} = 0 \quad (11)$$

with the boundary conditions (10).

We assume that the solution of (11) can be expressed by a set of base functions $\{1, x, x^2, \ldots\}$ in the form

$$u(x) = \sum_{i=0}^{+\infty} d_i x^i, \quad (12)$$

where d_i are coefficients to be determined. This provides us with the so-called rule of solution expression, i. e., the solution of (11) must be expressed in the same form as (12) and the other expressions must be avoided. Under the rule of solution expression denoted by (12), it is obvious to choose the auxiliary linear operator

$$\mathcal{L}[\phi(x;p)] = \frac{\partial \phi^2(x;p)}{\partial x^2}$$

with the property

$$\mathcal{L}[c_1 + c_2 x] = 0,$$

where c_1 and c_2 are constants. From (11), we define the nonlinear operators

$$\mathcal{N}[\phi(x;p)] = \frac{\partial \phi^2(x;p)}{\partial x^2} - \frac{\partial \phi^2(x;p)}{\partial x^2} \phi(x;p) - \frac{1}{2} \left(\frac{\partial \phi(x;p)}{\partial x} \right)^2 - \frac{1}{2}.$$

According to boundary conditions (10) and the rule of solution expression (12), it is straightforward that the initial approximations should be in the form $u_0(x) = -\frac{1}{2}x$, and we have the zero-order deformation equation (5) with the initial conditions

$$\phi(0;p) = 0, \quad \phi(1;p) = -0.5.$$

From (8) and (11), we have

$$R_m(\overrightarrow{u}_{m-1}) = u''_{m-1} - \sum_{i=0}^{m-1} u_i u''_{m-1-i} - \frac{1}{2} \sum_{i=0}^{m-1} u'_i u'_{m-1-i} - \frac{1}{2}(1 - \chi_m),$$

where the prime denotes differentiation with respect to the similarity variable x. Now, the solution of the mth-order deformation equation (7) for $m \geq 1$ becomes

$$u_m(x) = \chi_m u_{m-1}(x) + \hbar \mathcal{L}^{-1}[\mathcal{H}(x) R_m(\vec{u}_{m-1})] + c_1 + c_2 x,$$

where the constants c_i are determined by the initial condition

$$u_m(0) = 0, \quad u_m(1) = 0.$$

According to the rule of solution expression denoted by (12) and from (7), the auxiliary function $\mathcal{H}(x)$ should be in the form $\mathcal{H}(x) = -x^k$, where k is an integer. It is found that, when $k \leq -1$, the solution of the high-order deformation equation (7) contains the terms $\ln(x)$ or $\frac{1}{x^s}$ $(s \geq 1)$, which incidentally disobey the rule of solution expression (12). When $k \geq 1$, the base x always disappears in the solution expression of the high-order deformation equation (7), so that the coefficient of the term x cannot be modified even if the order of approximation tends to infinity. Thus, we have to set $k = 0$, which uniquely determines the corresponding auxiliary function $\mathcal{H}(x) = -1$.

Accordingly, the Nth-order approximate series solution $Y_N(x) = \sum_{i=0}^{N} u_i(x)$ can be obtained as follows:

$$\begin{aligned}
Y_1(x) &= \Big(-\frac{1}{2} - \frac{5}{16}h\Big)x + \frac{5}{16}hx^2,\\
Y_2(x) &= \Big(-\frac{1}{2} + \frac{65}{192}h^2 - \frac{5}{8}h\Big)x + \Big(-\frac{15}{64}h^2 + \frac{5}{8}h\Big)x^2 \\
&\quad -\frac{5}{48}h^2x^3,\\
Y_3(x) &= \frac{25}{24576}h^3 + \Big(-\frac{1}{2} + \frac{65}{96}h^2 - \frac{25}{3072}h^3 - \frac{15}{16}h \\
&\quad -\frac{5}{24576}h^2(1749h - 1664)\Big)x \\
&\quad +\Big[-h\Big(-\frac{535}{3072}h^2 + \frac{15}{64}h\Big) + \frac{15}{16}h - \frac{15}{32}h^2\Big]x^2 \\
&\quad +\Big[-\frac{5}{24}h^2 - h\Big(\frac{5}{48}h - \frac{15}{128}h^2\Big)\Big]x^3 + \frac{55}{768}h^3x^4.
\end{aligned}$$

It is obvious from Fig. 1 that to adjust and control the convergence region of solution series, the auxiliary parameter $\hbar$ should be chosen as

$$\hbar = 0.7.$$

The approximate series solution with 10 terms and $\hbar = 0.7$ is as follows:

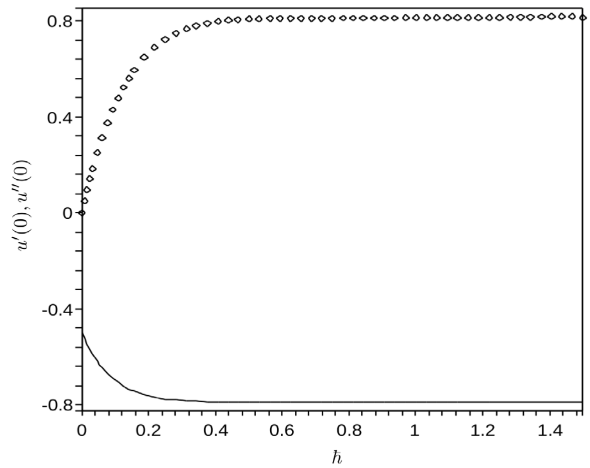

Fig. 1. $\hbar$-Curve of the 10th-order approximation of Example 1; solid line, $u'(0)$; symbols, $u''(0)$.

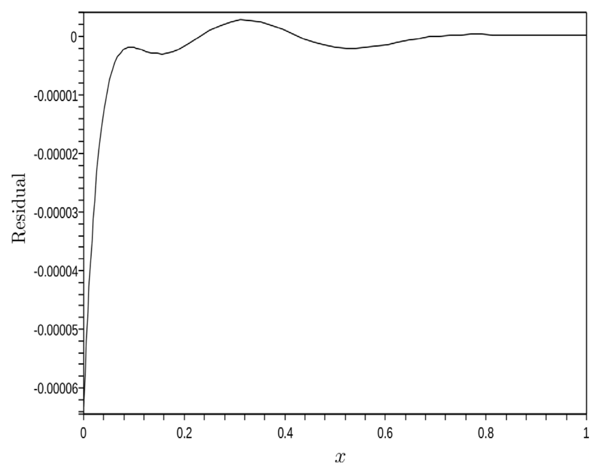

Fig. 2. Residual of the 10th-order approximation of Example 1.

$$\begin{aligned}
\sum_{m=0}^{10} u_m(x) &= 0.0005263196634 - 0.7872055314x \\
&\quad + 0.4051047674x^2 - 0.2123464565x^3 \\
&\quad + 0.1775735068x^4 - 0.1606474448x^5 \\
&\quad + 0.1407462143x^6 - 0.1057503008x^7 \\
&\quad + 0.06158360841x^8 - 0.02518533930x^9 \\
&\quad + 0.006330243171x^{10} \\
&\quad - 0.0007295855532x^{11}.
\end{aligned}$$

The approximate solution $\sum_{m=0}^{10} u_m(x)$ is of remarkable accuracy. The residual is shown in Figure 2.

4.2. Example 2

Consider the following variational problem [1, 4, 7]

$$\min \; v[y] = \int_0^1 \frac{1+y^2(x)}{y'^2(x)} \mathrm{d}x$$

with the boundary conditions

$$y(0) = 0, \quad y(1) = 0.5. \tag{13}$$

In this case the Euler-Lagrange equation is in the form

$$y'' - y''y^2 - yy'^2 = 0 \tag{14}$$

with the boundary conditions (13). We assume that the solution of (14) can be expressed by a set of base functions $\{x, x^3, x^5, \ldots\}$ in the form

$$u(x) = \sum_{i=0}^{+\infty} d_i x^{2i+1}, \tag{15}$$

where d_i are coefficients to be determined. In this example, the linear operator $\mathcal{L}$ is chosen as in Example 1. From (14), we define the nonlinear operators

$$\mathcal{N}[\phi(x;p)] = \frac{\partial\phi^2(x;p)}{\partial x^2} - \frac{\partial\phi^2(x;p)}{\partial x^2}\phi^2(x;p) - \phi(x;p)\left(\frac{\partial\phi(x;p)}{\partial x}\right)^2.$$

According to boundary conditions (13) and the rule of solution expression (15), it is straightforward that the initial approximations should be in the form $u_0(x) = \frac{1}{2}x$, and we have the zero-order deformation equation (5) with the initial conditions

$$\phi(0;p) = 0, \quad \phi(1;p) = 0.5.$$

From (8) and (14), we have

$$R_m(\overrightarrow{u}_{m-1}) = u''_{m-1} + \sum_{i=0}^{m-1} u''_i \sum_{j=0}^{m-1-i} u_j u_{m-1-i-j} - \sum_{i=0}^{m-1} u_i \sum_{j=0}^{m-1-i} u'_j u'_{m-1-i-j},$$

where the prime denotes differentiation with respect to the similarity variable x. Now, the solution of the mth-order deformation equation (7) for $m \geq 1$ becomes

$$u_m(x) = \chi_m u_{m-1}(x) + \hbar\mathcal{L}^{-1}[\mathcal{H}(x)R_m(\vec{u}_{m-1})] + c_1 + c_2 x,$$

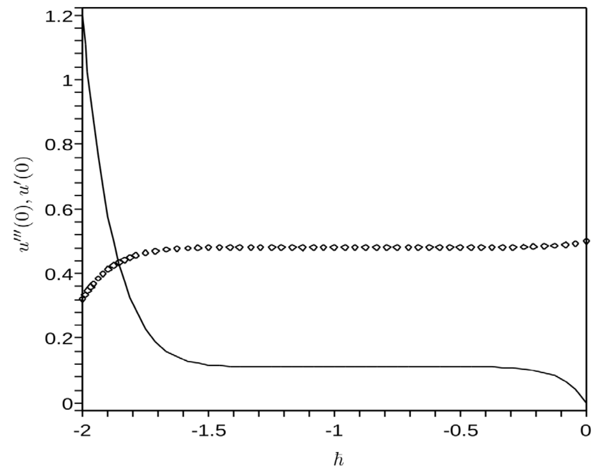

Fig. 3. $\hbar$-Curve of the 10th-order approximation of Example 2; solid line, $u'''(0)$; symbols, $u'(0)$.

where the constants c_i are determined by the initial condition

$$u_m(0) = 0, \quad u_m(1) = 0.$$

As in Example 1, in this example, we have

$$\mathcal{H}(x) = 1.$$

Accordingly, the Nth-order approximate series solution $Y_N(x) = \sum_{i=0}^{N} u_i(x)$ can be obtained as follows:

$$Y_1(x) = \left(\frac{1}{2} + \frac{1}{48}h\right)x - \frac{1}{48}hx^3,$$

$$Y_2(x) = \left(\frac{1}{2} + \frac{89}{3840}h^2 + \frac{1}{24}h\right)x + \left(-\frac{3}{128}h^2 - \frac{1}{24}h\right)x^3 + \frac{1}{3840}h^2x^5,$$

$$Y_3(x) = \left[\frac{1}{2} + \frac{89}{1920}h^2 + \frac{1}{16}h + \frac{1}{215040}h^2(5563h + 4984)\right]x + \left[-\frac{3}{64}h^2 + h\left(-\frac{2437}{92160}h^2 - \frac{3}{128}h\right) - \frac{1}{16}h\right]x^3 + \left[\frac{1}{1920}h^2 + h\left(\frac{53}{92160}h^2 + \frac{1}{3840}h\right)\right]x^5 - \frac{1}{645120}h^3x^7.$$

It is obvious from Fig. 3 that to adjust and control the convergence region of solution series, the auxiliary parameter $\hbar$ should be chosen as

$$\hbar = -1.$$

The approximate series solution with 10 terms and $\hbar=-1$ is as follows:

$$\sum_{m=0}^{10} u_m(x) = \frac{240566555447494721}{499918215976058880}x + \frac{256544305776974273}{1381352965197004800\,0}x^3 + \frac{56161532713337}{261180182495232000}x^5 + \frac{58525263353783}{4936305449159884800\,0}x^7 + \frac{9648392773}{2531438691876864000}x^9 + \frac{416051}{51662014119936000}x^{11} + \frac{575249}{4936305449159884800\,0}x^{13} + \frac{241}{1645435149719961600\,0}x^{15} + \frac{1}{1598422716870819840\,00}x^{17} + \frac{1}{6377706640314571161\,6000}x^{19}.$$

The approximate solution $\sum_{m=0}^{10} u_m(x)$ is of remarkable accuracy. The residual is shown in Figure 4.

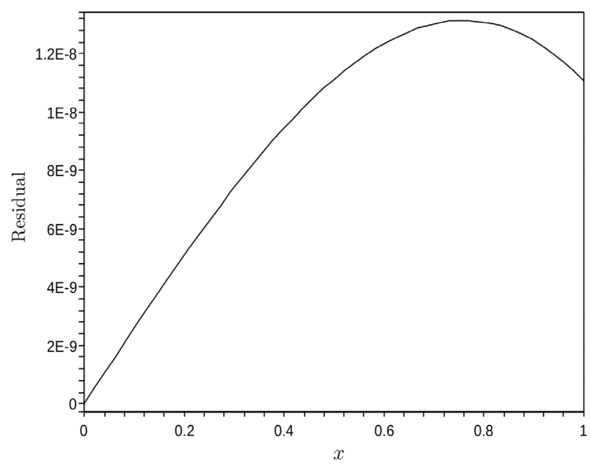

Fig. 4. Residual of the 10th-order approximation of Example 2.

5. Conclusions

The ordinary differential equations which arise from problems of the calculus of variations usually are nonlinear and are often difficult to analytically estimate. In the present paper the homotopy analysis method was applied to solve such problems. From the residual it was obvious that our results are in good agreement with the exact solution. In this regard the homotopy analysis method is found to be a very useful analytical technique to get highly accurate and purely analytic solutions of such kind of problems.

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